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THE METHOD OF POTENTIAL FUNCTIONS IN PROBLEMS OF THE THEORY OF ELASTICITY FOR BODIES WITH A DEFECT†

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The method of potential functions using a Fourier transformation in the class of slowly increasing distributions, corresponding to the classical method of complex potentials, is proposed for solving well-known problems of the theory of elasticity for bodies with a defect. It is shown that when a Fourier transformation with respect to all the spatial variables is used, the solution of the dynamic problem of the theory of elasticity can also be represented in terms of a jump in the stresses and displacements at the defect. The correctness of the transformed problem is considered (in terms of an analogue of the Lopatinskii condition). The solution of the system of Helmholtz equations, to which the system of Lamé equations is reduced in the case of the two-dimensional dynamic problem, is expressed in terms of the jump in the stresses and displacements at the defect as a result of solving the corresponding singular integral equations. © 2002 Elsevier Science Ltd. All rights reserved.

1. THE ANISOTROPIC THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY

Consider harmonic oscillations of a uniform anisotropic elastic half-space $\{x_3 > 0\}$ when there are no volume forces. In this case, the dynamic equations in displacements, written in a fixed rectangular Cartesian system of coordinates $x = (x_1, x_2, x_3)$, have the form (see, for example, [1])

$$\sum_{k,l,j=1}^{3} C_{ijkl}\frac{\partial^2 u_k}{\partial x_l \partial x_j} + \rho\nu^2 u_i = 0, \quad i = 1, 2, 3 \tag{1.1}$$

where u_i are the components of the complex displacement vector $\mathbf{u}(\mathbf{x})$ ($\bar{\mathbf{u}}(\mathbf{x}, t) = \mathrm{Re}\{\mathbf{u}(\mathbf{x})e^{-i\nu t}\}$, ρ is the density of the medium, C_{ijkl} $(i, j, k, l = 1, 2, 3)$ are the constants of elasticity, which satisfy the relations $C_{ijkl} = C_{jikl} = C_{klij}$, and the time factor $e^{-i\nu t}$ is omitted everywhere.

The solution $\mathbf{u} = u(u_1, u_2, u_3)$ of Eq. (1.1) when $x_3 > 0$ $(x_3 < 0)$ will be said to be departing from the plane $\{x_3 = 0\}$ into the half-space $\{x_3 > 0\}$ $(\{x_3 < 0\})$, if $u_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ are slow growth distributions and, in this case

$$\operatorname{supp} u_i(x_1, x_2, x_3) \subset \{x_3 > 0\}(\{x_3 < 0\}), \quad i = 1, 2, 3 \tag{1.2}$$

$$\operatorname{sing\,supp} U_i(\xi_1, \xi_2, \xi_3) \cap \{\xi_3 < 0\}(\{\xi_3 > 0\}) = 0, \quad i = 1, 2, 3 \tag{1.3}$$

Here and henceforth Fourier transforms of the corresponding functions will be denoted by capital letters.

To find the solution of Eq. (1.1) in the half-space $\{x_3 > 0\}$ in the class of solutions departing from the plane $\{x_3 = 0\}$ into the half-space $\{x_3 > 0\}$, which satisfy the conditions

$$\begin{gathered}\sum_{j=1}^{3} \alpha_j^i(x_1, x_2)\frac{\partial u_i}{\partial x_j}(x_1, x_2, 0) = g^i(x_1, x_2) \\ u_i(x_1, x_2, 0) = h^i(x_1, x_2); \quad x_i \in R, \quad i = 1, 2, 3\end{gathered} \tag{1.4}$$

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where $\alpha_j^i(\cdot,\cdot), g^i(\cdot,\cdot), h^i(\cdot,\cdot)$ $(i,j = 1, 2, 3)$ are specified functions, we will apply a Fourier transformation with respect to the variables x_1, x_2, x_3 to Eqs (1.1), taking conditions (1.2) into account and using boundary conditions (1.4). The Fourier transforms of the distributions $u_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ then satisfy the equations

$$\sum_{k,l,j=1}^{3} C_{ijkl}\xi_l\xi_j U_k(\xi_1,\xi_2,\xi_3) - \rho v^2 U_i(\xi_1,\xi_2,\xi_3) = F_i(\xi_1,\xi_2,\xi_3), \quad i = 1,2,3 \tag{1.5}$$

where

$$F_i(\xi_1,\xi_2,\xi_3) = \frac{i}{\sqrt{\pi}}\sum_{k=1}^{3}\left[\sum_{j=1}^{2}(C_{ijk3} + C_{i3kj})\xi_j U_k^0(\xi_1,\xi_2) + \right.$$

$$\left. + C_{i3k3}(iU_k^1(\xi_1,\xi_2) + \xi_3 U_k^0(\xi_1,\xi_3))\right], \quad i = 1,2,3$$

and $U_k^0(\xi_1, \xi_2)$, $U_k^1(\xi_1, \xi_2)$ $(k = 1, 2, 3)$ are the Fourier transforms of the functions $u_i(x_1, x_2, 0)$, $\partial u_i/\partial x_3(x_1, x_2, 0)$ $(i = 1, 2, 3)$, found from boundary conditions (1.4) after a Fourier transformation with respect to the variables x_1, x_2.

For convenience we will rewrite system of equations (1.4) in the form $\mathbf{AU} = \mathbf{F}$ and denote by

$$\Delta(\xi_1,\xi_2,\xi_3) = \det \mathbf{A}, \quad \Delta_i(\xi_1,\xi_2,\xi_3), \quad i = 1,2,3$$

the determinant of the matrix obtained from $\mathbf{A}$ by replacing the i-th column by $(F_1, F_2, F_3)^T$. It can be shown that when $\operatorname{Im}\xi_1 = \operatorname{Im}\xi_2 = \operatorname{Im} v = 0$ the equation $\Delta(\xi_1, \xi_2, \xi_3) = 0$ has k pairs of complex-conjugate and 6–2k real roots ξ_3^j $(k = 0, 1, 2, 3, j = 1, 2, \ldots, 6)$. The Fourier transforms of the required distributions are given by the equations

$$U_i(\xi_1,\xi_2,\xi_3) = \frac{\Delta_i(\xi_1,\xi_2,\xi_3)}{\Delta(\xi_1,\xi_2,\xi_3)}, \quad i = 1,2,3$$

where

$$\Delta_i(\xi_1,\xi_2,\xi_3) = 0, \quad i = 1,2,3, \quad \text{when} \quad \xi_3 = \xi_3^j, \ \operatorname{Im}\xi_3^j > 0, \ j = 1,2,\ldots,k$$

We therefore have $3k$ additional conditions in the case of k roots ξ_3 with the positive imaginary part of the equation $\Delta(\xi_1, \xi_2, \xi_3) = 0$, and $15 - 3k$ independent coefficients remain in boundary conditions (1.4).

Hence, depending on the number of roots ξ_3 of the equation $\Delta(\xi_1, \xi_2, \xi_3) = 0$ with positive imaginary part, the number of additional conditions in the boundary-value problems may be different.

In the case considered previously [2] when solving the boundary-value problem

$$P(D)\mathbf{u} = f(\mathbf{x}), \quad x_n \geqslant 0, \quad n \geqslant 3$$

$$B_j(D)\mathbf{u} = g_j(\mathbf{x}), \quad x_n = 0, \quad j = 1,\ldots,\mu$$

where $x = (x_1, \ldots, x_n)$, $P(\xi)$ was a homogeneous elliptic polynomial of order m and the number of boundary conditions was identical with the number of roots λ of the equation $P(\xi', \lambda) = 0$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$, in which the imaginary part is positive, the order of the differential operator $B_j(D)$ does not exceed $m - 1$.

Note that the difference between the proposed approach and the method considered previously in [2], lies in the fact that the Fourier transformation is not carried out with respect to the variable x_n in [2]. Hence, the number of boundary conditions in [2], written for the required function, corresponds to the number of additional conditions introduced in the present paper, written for the Fourier transforms of the required function. Boundary conditions (1.4) in the variables x_1, x_2, x_3 can be assumed to be analogous to the Lopatinskii condition with variables $\xi_1, \ldots, \xi_n$. In both cases these conditions ensure that the problem is correct. For example, if there are less coefficients in boundary conditions (1.4), the problem may turn out to be overdetermined.

The splitting of the conditions into boundary and additional conditions (Lopatinskii and boundary conditions) is made solely for convenience and to simplify the calculations: they can be combined in a natural way by changing to the variables x_1, x_2, x_3 $(x_1, \ldots, x_n)$ or to the variables ξ_1, ξ_2, ξ_3 $(\xi_1, \ldots, \xi_n)$.

These conditions impose certain limitations on the coefficients $\alpha_j^i(\cdot,\cdot)$, $h^i(\cdot,\cdot)$, $g^i(\cdot,\cdot)$ $(i, j = 1, 2, 3)$ (the operators $B_j(D)$ $(j = 1, \ldots, \mu)$ in [2]). Hence, the approaches considered to solving the problem in a half-space are equivalent ($n = 3$ in [2]).

If the roots ξ_3 of the equation $\Delta(\xi_1, \xi_2, \xi_3) = 0$ with positive imaginary part are known, the solution of problem (1.1), (1.3) can be written explicitly. It is then easy to obtain representations of the solutions of the problems in terms of the jumps in the stresses and displacements in the $\{x_3 = 0\}$ plane and it is convenient to investigate, for example, problems of the diffraction of an elastic harmonic wave by a sealed-in rigid screen situated in the $\{x_3 = 0\}$ plane and also the problem of the propagation of a crack in the $\{x_3 = 0\}$ plane [1].

2. FORMULATIONS OF THE PROBLEMS FOR AN ISOTROPIC ELASTIC PLANE WITH A DEFECT

Suppose an infinitely thin defect $\Gamma = \{z = 0, \alpha < x < \beta\}$ is situated in a uniform isotropic elastic space. We will consider two-dimensional problems of the dynamic theory of elasticity when $\partial/\partial y = 0$. We will assume that the stresses $\tau_{xz}(\cdot,\cdot)$, $\sigma_z(\cdot,\cdot)$ and the displacements $u(\cdot,\cdot)$, $v(\cdot,\cdot)$ depend harmonically on time, and there are no volume forces. We will seek the complex amplitudes of the functions, omitting the time factor e^{-ikt}.

As is well known, with the above assumptions the equations of the dynamic theory of elasticity in displacements have the form

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 v}{\partial x\partial z}+\mu\frac{\partial^2 u}{\partial z^2}+\rho k^2 u=0$$

$$\mu\frac{\partial^2 v}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 u}{\partial x\partial z}+(\lambda+2\mu)\frac{\partial^2 v}{\partial z^2}+\rho k^2 v=0 \quad (x,z)\in R^2\setminus\bar{\Gamma} \tag{2.1}$$

where λ and μ are the Lamé constants and ρ is the density of the body.

The solution $u(x, z)$, $v(x, z)$ of the Lamé equations (2.1) when $z > 0$ $(z < 0)$ will be said to depart from the straight line $z = 0$ in the half-plane $\{z > 0\}$ $(\{z < 0\})$ [3], if $u(x, z)$, $v(x, z)$ are the distributions of slow growth and

$$\operatorname{supp} u(x,z),\ \operatorname{supp} v(x,z)\subset\{z>0\}(\{z<0\}) \tag{2.2}$$

$$\operatorname{sing\,supp} \dot{U}(\xi,\ \zeta)\cap\{\zeta<0\}(\{\zeta>0\})=0$$

$$\operatorname{sing\,supp} V(\xi,\ \zeta)\cap\{\zeta<0\}(\{\zeta>0\})=0 \tag{2.3}$$

We will seek solutions of the problem in the class $H^1_{loc}(R^2)\cap\tilde{S}'$ ($\tilde{S}'$ is the class of solutions departing from the straight line $z = 0$). If the values of the function $f(\cdot)\in\tilde{S}'$, defined when $z > 0$ and $z < 0$, are identical when $z = 0 - 0$ and $z = 0 + 0$, it is natural to supplement it when $z = 0$ by assuming $f(0) = f(0 - 0) \equiv f(0 + 0)$. It can be shown that Eqs (2.3) includes conditions at infinity; functions from the class $\tilde{S}'$ carry away energy at infinity. The fact that the displacements belong to the class $H^1_{loc}(R^2)$ ensures that the energy is finite in any bounded region from R^2. The boundary conditions of specific problems will be considered in Section 4.

As is well known, the displacements $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ are determined by the longitudinal and transverse potentials $\varphi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$

$$u=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial z},\quad v=\frac{\partial\varphi}{\partial z}-\frac{\partial\psi}{\partial x} \tag{2.4}$$

Instead of the Lamé equations for the functions $u(\cdot,\cdot)$, $v(\cdot,\cdot)$ we have the Helmholtz equations for the potentials $\varphi(\cdot,\cdot)$, $\psi(\cdot,\cdot)$

$$\Delta\varphi+k_1^2\varphi=0,\quad \Delta\psi+k_2^2\psi=0\ \ (x,z)\in R^2\setminus\bar{\Gamma};\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2} \tag{2.5}$$

$$k_i=k/c_i,\quad i=1,2;\quad c_1=\sqrt{(\lambda+2\mu)/\rho},\quad c_2=\sqrt{\mu/\rho}$$

where c_1 and c_2 are the propagation velocities of longitudinal and transverse waves in a uniform isotropic elastic medium.

The equations for the new required functions $\varphi(\cdot,\cdot)$, $\psi(\cdot,\cdot)$ can be split. However, if the longitudinal and transverse potentials are functions from the class $H^1_{loc}(R^2) \cap \tilde{S}'$, the displacements, according to relations (2.4), may turn out to be less smooth functions.

Using relations (2.4) and the equation in the sense of generalized functions from the class $\tilde{S}'$ of mixed derivatives $\partial^2/\partial x\partial z = \partial^2/\partial z\partial x$ for the functions $\varphi(\cdot,\cdot)$, $\psi(\cdot,\cdot)$, $u(\cdot,\cdot)$, $v(\cdot,\cdot)$, it can be shown that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=\Delta\varphi,\quad \frac{\partial u}{\partial z}-\frac{\partial v}{\partial z}=\Delta\psi \quad (x,z)\in R^2\setminus\Lambda$$

where

$$\Lambda=\{(x,z): x\in R, z=0\}$$

Then

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=-k_1^2\varphi,\quad \frac{\partial u}{\partial z}-\frac{\partial v}{\partial x}=-k_2^2\psi \quad (x,z)\in R^2\setminus\Lambda \tag{2.6}$$

and the following equations hold

$$\Delta u=-k_1^2\frac{\partial\varphi}{\partial x}-k_2^2\frac{\partial\psi}{\partial z},\quad \Delta v=k_2^2\frac{\partial\psi}{\partial x}-k_1^2\frac{\partial\varphi}{\partial z} \quad (x,z)\in R^2\setminus\Lambda \tag{2.7}$$

Note that Eqs (2.6) and (2.7) only hold when $z>0$ and when $z<0$, where the functions $\varphi(\cdot,\cdot)$, $\psi(\cdot,\cdot)$, $u(\cdot,\cdot)$, $v(\cdot,\cdot)$ and their first derivatives are understood in the usual sense, i.e. they are considered in those classes to which they actually belong.

3. REPRESENTATIONS OF THE SOLUTIONS OF THE PROBLEMS IN TERMS OF JUMPS IN THE STRESSES AND DISPLACEMENTS

We will consider the auxiliary problem of a jump. We will seek solutions $\varphi(\cdot,\cdot)$, $\psi(\cdot,\cdot)$ of Helmholtz equations (2.5) in the region $R^2\setminus\Lambda$. For the present we will write the boundary conditions in terms of the stresses and displacements

$$[u]|_\Lambda=a_u(x),\quad [v]|_\Lambda=a_v(x),\quad [\tau_{xz}]|_\Lambda=a_\tau(x),\quad [\sigma_z]|_\Lambda=a_\sigma(x);\quad x\in R \tag{3.1}$$

$$[f]|_\Lambda=f(x,0+0)-f(x,0-0)\equiv f^+(x,0)-f^-(x,0)$$

In relations (3.1) the functions on the right-hand sides of the equations will be called potential functions. Applying a Fourier transformation with respect to the variable x to Eqs (2.4), we obtain

$$U(\xi,z)=-i\xi\Phi(\xi,z)+\frac{\partial\Phi}{\partial z}(\xi,z),\quad V(\xi,z)=\frac{\partial\Phi}{\partial z}(\xi,z)+i\xi\Psi(\xi,z) \tag{3.2}$$

Hence the first two equations of (3.1) can easily be written for the Fourier transforms of the functions $\varphi(\cdot,\cdot)$, $\psi(\cdot,\cdot)$ and for their first derivatives with respect to z.

We will transform the remaining two conditions in (3.1). Using the well-known formulae

$$\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\right),\quad \sigma_z=\lambda\frac{\partial u}{\partial x}+(\lambda+2\mu)\frac{\partial v}{\partial z}$$

and Eqs (2.6) we have

$$\tau_{xz}=2\mu\frac{\partial v}{\partial x}-\chi_2\psi,\quad \sigma_z=-2\mu\frac{\partial u}{\partial x}-\chi_1\varphi;\quad \chi_1=k_1^2(\lambda+2\mu),\quad \chi_2=k_2^2\mu$$

We apply a Fourier transformation with respect to x to these equations

$$T_{xz}(\xi, z) = -i2\mu\xi\frac{\partial\Phi}{\partial z}(\xi, z) + (2\mu\xi^2 - \chi_2)\Psi(\xi, z)$$

$$\Sigma_z(\xi, z) = (2\mu\xi^2 - \chi_1)\Phi(\xi, z) + i2\mu\xi\frac{\partial\Psi}{\partial z}(\xi, z) \tag{3.3}$$

Then $U(\xi, z)$, $V(\xi, z)$, $T_{xz}(\xi, z)$, $\Sigma_z(\xi, z)$ can be expressed in terms of the functions $\Phi(\xi, z)$, $\Psi(\xi, z)$ and their first derivatives with respect to the variable z. We will denote by $\Phi_0(\xi)$, $\Phi_1(\xi)$, $\Psi_0(\xi)$, $\Psi_1(\xi)$ the values of the functions $\Phi(\xi, z)$, $\partial\Phi/\partial z(\xi, z)$, $\Psi(\xi, z)$, $\partial\Psi/\partial z(\xi, z)$ where $z = 0$. As a result, boundary conditions (3.1) take the following form in terms of the Fourier transforms of the functions

$$[\Psi_1 - i\xi\Phi_0]|_{\tilde{\Lambda}} = A_u(\xi), \quad [\Phi_1 + i\xi\Psi_0]|_{\tilde{\Lambda}} = A_v(\xi)$$

$$[(2\mu\xi^2 - \chi_2)\Psi_0 - i2\xi\Phi_1]|_{\tilde{\Lambda}} = A_\tau(\xi) \tag{3.4}$$

$$[(2\mu\xi^2 - \chi_1)\Phi_0 + i2\mu\xi\Psi_1]|_{\tilde{\Lambda}} = A_\sigma(\xi); \quad \xi \in R$$

$$\tilde{\Lambda} = \{(\xi, z): \ \xi \in R, \ z = 0\}, \quad [F]|_{\tilde{\Lambda}} = F^+(\xi) - F^-(\xi)$$

Here $F^{\pm}(\xi)$ are the Fourier transforms of the functions $f^{\pm}(x, 0)$ from (3.1).

To solve this problem we now apply a Fourier transformation with respect to the variables x and z in Helmholtz equations (2.5)

$$(k_1^2 - \xi^2 - \zeta^2)\Phi^{\pm}(\xi, \zeta) = \pm\frac{1}{\sqrt{2\pi}}[\Phi_1^{\pm}(\xi) - i\zeta\Phi_0^{\pm}(\xi)]$$

$$(k_2^2 - \xi^2 - \zeta^2)\Psi^{\pm}(\xi, \zeta) = \pm\frac{1}{\sqrt{2\pi}}[\Psi_1^{\pm}(\xi) - i\zeta\Psi_0^{\pm}(\xi)] \tag{3.5}$$

The distributions $\Phi^+(\xi, \zeta)$ ($\Psi^+(\xi, \zeta)$) and $\Phi^-(\xi, \zeta)$ ($\Psi^-(\xi, \zeta)$) are Fourier transforms of the function $\varphi(x, z)$ ($\psi(x, z)$) when $z > 0$ and $z < 0$.

Conditions (2.2) for the required functions are satisfied if and only if the following equalities hold

$$\Phi_1^{\pm}(\xi) \mp i\gamma_1(\xi)\Phi_0^{\pm}(\xi) = 0, \quad |\xi| > k_1 \tag{3.6}$$

$$\Psi_1^{\pm}(\xi) \mp i\gamma_2(\xi)\Psi_0^{\pm}(\xi) = 0, \quad |\xi| > k_2 \tag{3.7}$$

where

$$\gamma_j(\xi) = \{|\xi| \geqslant k_j: \ +i\sqrt{\xi^2 - k_j^2}; \quad |\xi| < k_j: \ -\sqrt{k_j^2 - \xi^2}\}, \quad j = 1, 2$$

This means that the distributions $\Phi^{\pm}(\xi, \zeta)$, $\Psi^{\pm}(\xi, \zeta)$ with respect to the variable ζ are limiting values of the functions, analytic in the upper half-plane. It can be shown that conditions (2.3) are equivalent to Eqs (3.6) when $|\xi| < k_1$, and (3.7) when $|\xi| < k_2$. Hence, the fact that the displacements belong to the class $\tilde{S}'$ is equivalent to satisfying Eqs (3.6) and (3.7) for all ξ. Then, to determine the functions $\Phi_0^{\pm}(\xi)$, $\Phi_1^{\pm}(\xi)$, $\Psi_0^{\pm}(\xi)$, $\Psi_1^{\pm}(\xi)$ we have the system of linear algebraic equations (3.4) and (3.6), (3.7) with $\xi \in R$. These functions can be expressed in terms of Fourier transforms of the known jumps in the stresses and displacements

$$\Phi_0^{\pm}(\xi) = \frac{1}{\chi\gamma_1(\xi)}(\mp i2\mu\xi p_{12}(\xi)A_u(\xi) + iq_1(\xi)A_v(\xi) - \chi_1\xi A_\tau(\xi) \pm p_{12}(\xi)A_\sigma(\xi))$$

$$\Phi_1^{\pm}(\xi) = \frac{1}{\chi}(2\mu\xi p_{12}(\xi)A_u(\xi) \mp q_1(\xi)A_v(\xi) \mp i\chi_1\xi A_\tau(\xi) + ip_{12}(\xi)A_\sigma(\xi))$$

$$\Psi_0^{\pm}(\xi) = \frac{1}{\chi\gamma_2(\xi)}(iq_2(\xi)A_u(\xi) \pm i2\mu\xi p_{21}(\xi)A_v(\xi) \pm p_{21}(\xi)A_\tau(\xi) + \chi_2\xi A_\sigma(\xi)) \tag{3.8}$$

$$\Psi_1^{\pm}(\xi) = \frac{1}{\chi}(\mp q_2(\xi)A_u(\xi) - 2\mu\xi p_{21}(\xi)A_v(\xi) + ip_{21}(\xi)A_\tau(\xi) \pm i\chi_2\xi A_\sigma(\xi))$$

Here

$$\chi=-2\chi_1\chi_2,\quad p_{ij}(\xi)=\chi_j\gamma_i(\xi),\quad q_j(\xi)=\chi_j(\chi_2-2\mu\xi^2);\quad i,j=1,2$$

We can now correctly determine the functions $\Phi^{\pm}(\xi,\zeta)$, $\Psi^{\pm}(\xi,\zeta)$ using Eqs (3.5) and (3.8). It is convenient to carry out an inverse Fourier transformation with respect to the variable ζ for these functions. It can be easily verified that

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{e^{-i\zeta z}}{\gamma_1(\xi)\pm\zeta}d\zeta=-ie^{\pm iz\gamma_1(\xi)}$$

Then

$$\Phi^{\pm}(\xi,z)=\Phi_0^{\pm}(\xi)e^{\pm iz\gamma_1(\xi)},\quad \Psi^{\pm}(\xi,z)=\Psi_0^{\pm}(\xi)e^{\pm iz\gamma_2(\xi)} \tag{3.9}$$

After the inverse Fourier transformation with respect to the variable ξ we obtain the functions $\varphi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$. The required functions $u(\cdot,\cdot)$, $v(\cdot,\cdot)$ can be obtained using Eqs (2.4). Hence, the solution of the initial problem can be expressed in terms of jumps in the stresses and displacements or potential functions.

Note that one can immediately obtain the Fourier transforms of the required functions $\tau_{xz}(\cdot,\cdot)$, $\sigma_z(\cdot,\cdot)$, $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ using formulae (3.2) and (3.3), if we know the Fourier transforms of the functions $\varphi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$. Dynamic problems can then be considered in terms of stresses and displacements.

4. THE INTEGRAL EQUATIONS OF THE TWO-DIMENSIONAL BOUNDARY-VALUE PROBLEMS

1. We will consider the plane problem of diffraction, when the defect is a soldered-on rigid screen. In this case, the displacement vector is continuous on passing through the straight line $z = 0$ and the boundary conditions of the problem have the form

$$u|_\Gamma=-u_0(x),\quad v|_\Gamma=-v_0(x);\quad x\in(\alpha,\beta) \tag{4.1}$$

$$[u]|_{\Lambda\setminus\bar\Gamma}=0,\quad [v]|_{\Lambda\setminus\bar\Gamma}=0,\quad [\tau_{xz}]|_{\Lambda\setminus\bar\Gamma}=0,\quad [\sigma_z]|_{\Lambda\setminus\bar\Gamma}=0 \tag{4.2}$$

where $u_0(\cdot)$, $v_0(\cdot)$ are functions specified in the interval (α,β). Using the representations of the solutions of the two-dimensional problems in terms of jumps in the stresses and displacements (see Section 3) it can be shown that the functions $a_u(\cdot)$ and $a_v(\cdot)$ are identically equal to zero, while the functions $a_\tau(\cdot)$ and $a_\sigma(\cdot)$ are equal to zero in the set $\Lambda\setminus\bar\Gamma$. Hence, to solve the diffraction problem it is necessary to obtain the functions $a_\tau(\cdot)$ and $a_\sigma(\cdot)$ at the defect. Using boundary conditions (4.1) and Eqs (3.2) and (3.9), we obtain integral equations for determining these functions

$$L_1a_\tau=\frac{i}{4\pi(\lambda+2\mu)}\int\limits_\alpha^\beta a_\tau(t)\int\limits_{-\infty}^{+\infty}F_{12}(\xi)e^{i(t-x)\xi}d\xi dt=u_0(x);\quad x\in(\alpha,\beta) \tag{4.3}$$

$$\frac{i}{4\pi\mu}\int\limits_\alpha^\beta a_\sigma(t)\int\limits_{-\infty}^{+\infty}F_{21}(\xi)e^{i(t-x)\xi}d\xi dt=v_0(x);\quad x\in(\alpha,\beta) \tag{4.4}$$

Here

$$F_{ij}(\xi)=\frac{1}{\gamma_i(\xi)}+\frac{1}{k_i^2}(\gamma_j(\xi)-\gamma_i(\xi))=F_{ij}^1|\xi|^{-1}+O(|\xi|^{-2})\ (\text{при}\ |\xi|\to\infty),\quad i,j=1,2$$

$$F_{12}^1=-i\frac{\lambda+3\mu}{2\mu},\quad F_{21}^1=-i\frac{\lambda+3\mu}{2(\lambda+2\mu)}$$

Note that previously [4, 5] integral equations of static problems of the plane theory of elasticity were obtained for isotropic bodies with defects; in that case the logarithmic singularities of the integral equations are contained in the integrals with a Cauchy kernel with a variable limit. When solving dynamic problems, integral equations (4.3) and (4.4) contain logarithmic singularities in integrals with infinite limits, and it is therefore convenient to consider them as pseudo-differential equations.

2. We will consider the two-dimensional dynamic problem when the defect is a crack. Then, on passing through the straight line $z = 0$, the stress vector remains continuous and, in addition to conditions (4.2), the following functions are given

$$\sigma_z|_\Gamma = -\sigma_z^0(x), \quad \tau_{xz}|_\Gamma = -\tau_{xz}^0(x); \quad x \in (\alpha, \beta)$$

In the problem of a jump, considered in Section 3, only the functions $a_u(\cdot)$ and $a_v(\cdot)$ are non-zero on Γ, and the integral equations equivalent to the boundary-value problem, have the form

$$L_2 a_u = \frac{i\mu}{4\pi}\int_\alpha^\beta a_u(t)\int_{-\infty}^{+\infty} G_{12}(\xi)e^{i(t-x)\xi}d\xi dt = -\tau_{xz}^0(x); \quad x \in (\alpha, \beta) \tag{4.5}$$

$$\frac{i\mu}{4\pi}\int_\alpha^\beta a_v(t)\int_{-\infty}^{+\infty} G_{21}(\xi)e^{i(t-x)\xi}d\xi dt = -\sigma_z^0(x); \quad x \in (\alpha, \beta) \tag{4.6}$$

Here

$$G_{ij}(\xi) = \frac{k_2^2}{\gamma_j(\xi)} + \frac{4}{k_2^2}\xi^2\left(\gamma_i(\xi) - \gamma_j(\xi) + \frac{k_1^2 - k_i^2}{\gamma_1(\xi)}\right) =$$

$$= G_{ij}^1|\xi| + O(1) \text{ (при } |\xi| \to \infty), \quad i, j = 1, 2$$

$$G_{12}^1 = G_{21}^1 = \frac{2(\lambda+\mu)}{\lambda+2\mu}$$

5. NUMERICAL SOLUTIONS OF THE INTEGRAL EQUATIONS

According to one of the approaches to solving integral equations of the first kind, we will analytically separate the principal parts of the operators of the left-hand of the integral equation and, in the numerical method, the action of the principal part will be taken into account in explicit form.

The solution $a_\tau(\cdot)$ of integral equation (4.3) will be sought in the space of the distributions $\tilde{H}^{-1/2}(\Gamma)$ [7]

$$\tilde{H}^s(\Gamma) = \{u : (1+|\xi|)^s U(\xi) \in L_2(R^1), \ supp\, u \subset \bar{\Gamma}\}$$

It can be shown [7, 2], that the left-hand side of integral equation (4.3) defines a bounded pseudo-differential operator L_1: $\tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $H^s(\Gamma)$ – the contraction of $H^s(R^1)$ on Γ. Moreover [8, 7], L_1 is a Fredholm pseudo-differential operator of order –1 and index 0. We will separate the principal part of the operator L_1 analytically. After this, we split the left-hand side of integral equation (4.3) into two terms: a singular term with $F_{12}^1|\xi|^{-1}$ instead of $F_{12}(\xi)$ and a regular term with $F_{12}^r(\xi) = F_{12}(\xi) - F_{12}^1|\xi|^{-1}$ instead of $F_{12}(\xi)$. Using generalized Parseval equalities and carrying out inverse Fourier transformations of the functions $F_{12}^1|\xi|^{-1}$ and $F_{12}^r(\xi)$ [9], we obtain an integral equation equivalent to integral equations (4.3), in which, for convenience, we have changed from the interval (α, β) to the interval (–1, 1)

$$\tilde{L}_1\varphi_1 = \int_{-1}^{1}\left(\ln\frac{1}{|t-x|} + K_1(t,x)\right)\varphi_1(t)dt = f_1(x); \quad x \in (-1, 1) \tag{5.1}$$

Here

$$\varphi_1(t) = a_\tau(\gamma_+ - \gamma_- t), \quad K_1(t,x) = \mu b\tilde{f}_{12}(\gamma_-(x-t)) - \ln\frac{1}{|t-x|}$$

$$\tilde{f}_{ij}(t) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{k_i^2|t|}\left(k_i H_1^{(2)}(k_i|t|) - k_j H_1^{(2)}(k_j|t|)\right) - H_0^{(2)}(k_i|t|)\right), \quad i = 1, 2$$

$$f_1(x) = -\frac{c}{\gamma_-} u_0(\gamma_+ - \gamma_- x), \quad \gamma_\pm = \frac{\alpha \pm \beta}{2} \tag{5.2}$$

$$b = -i\sqrt{\frac{\pi}{2}}\frac{1}{\lambda + 3\mu}, \quad c = -\frac{2\pi\mu(\lambda + 2\mu)}{\lambda + 3\mu}$$

and $H_n^{(2)}(\cdot)$ is the Hankel function of the second kind of order n.

For a numerical solution of the integral equation obtained we used the Bubnov-Galerkin method with Chebyshev polynomials of the first kind as basis and trial functions. The numerical method is stable and effective for "small" dimensions of the screen [7] (the screen is assumed to be small if the length Γ is small compared with the wavelengths in space, i.e. when $k_i(\beta-\alpha) \ll 1$ $(i = 1, 2)$). As a rule, a few Chebyshev polynomials are sufficient (for example, 5–10, depending on the required accuracy) for a good approximation of the function $\varphi_1(\cdot)$. This is due to the fact that the Chebyshev polynomials are eigenfunctions of the principal part of the operator L_1. Moreover, the introduction of weighting factors in the case of small screens enables us to take into account the behaviour of the solution in the neighbourhood of end points, which on the whole, determine the solution on the screen. This method, as numerical experiments show, is effective up to values of $k_i(\beta-\alpha) \sim 1$ $(i = 1, 2)$.

In Fig. 1 curve 1 shows the approximate solution of integral equation (5.1) in the case when a plane wave is incident on the screen at an angle of $\pi/4$ with $\alpha = 0.01$, $\beta = 0.02$, $k = 0.01$, $\lambda = 0.5$, $\mu = 0.5$, $\rho = 2700$ and $M = N = 10$; N is the number of Chebyshev polynomials in the expansion of the required function and M is the number of nodes in Hermite's quadrature formula used for the auxiliary calculations [4]. In view of the symmetry, we only show the region $t \geqslant 0$.

We will seek a solution $a_\sigma(\cdot)$ of integral equation (4.4) in the space of the distributions $\tilde{H}^{-1/2}(\Gamma)$. As in the case of integral equation (4.3) the left-hand side of integral equation (4.4) will be represented by a bounded Fredholm pseudo-differential operator of order -1 and index 0, acting from $\tilde{H}^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. It can then be shown [7] that the functions $u(\cdot,\cdot)$, $v(\cdot,\cdot) \in H^1_{\mathrm{loc}}(R^2)$ for any $a_\tau(\cdot)$, $a_\sigma(\cdot) \in \tilde{H}^{-1/2}(\Gamma)$. Hence, the functions $u(\cdot,\cdot)$, $v(\cdot,\cdot)$ satisfy the "condition on the edge", and the problem of diffraction by a soldered-in rigid screen is uniquely solvable [6].

To regularize an integral equation of the first kind analytically we separate the principal part of the operator on the left-hand side. The following integral equation equivalent to (4.4) can be obtained

$$\int_\alpha^\beta \ln|t-x|\, a_\sigma(t)dt + \int_\alpha^\beta ((\lambda+2\mu)b\tilde{f}_{21}(t-x) - \ln|t-x|)a_\sigma(t)dt = cv_0(x); \quad x \in (\alpha, \beta) \tag{5.3}$$

The function $\tilde{f}_{21}(\cdot)$ and the constants b and c are determined by the third and last two formulae of (5.2).

In Fig. 1 curve 2 shows the approximate solution of the integral equation to which integral equation (5.3) is reduced when changing from the interval (α, β) to the interval $(-1, 1)$. We will denote the required function by $\varphi_1(\cdot)$.

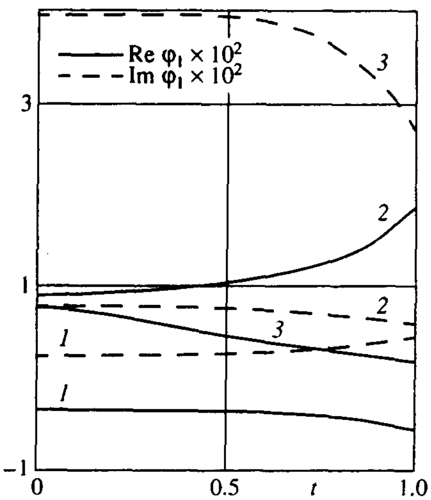

Fig. 1

Hence, we have considered integral equations with a logarithmic singularity in the kernel, equivalent to the two-dimensional problem of the diffraction of an elastic harmonic wave by a soldered-in rigid screen.

Consider integral equation (4.5). It can be shown [7, 8], that the operator L_2: $\tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ of the left-hand side of integral equation (4.5) is a bounded Fredholm pseudo-differential operator of order +1 and index zero. As in the case of integral equations (4.3) and (4.4), we separate the principal part of the operator L_2. For convenience we split the kernel of the operator into regular and singular parts. When calculating the inverse Fourier transformation of the function $G_{12}(\cdot)$ we will use the equation

$$G_{12}(\xi) = \frac{k_2^2}{\gamma_2(\xi)} + \frac{4}{k_2^2}(k_1^2\gamma_1(\xi) - k_2^2\gamma_2(\xi) - \gamma_1^3(\xi) + \gamma_2^3(\xi))$$

Then, using the generalized Parseval equalities and calculating the inverse Fourier transformation for the regular and singular parts of the kernel of the integral operator [9], we obtain an integral equation in which, using replacement of variables, we change from the interval (α, β) to the interval $(-1, 1)$

$$\tilde{L}_2\varphi_2 = \frac{\partial}{\partial x}\int_{-1}^{1}\left(\frac{\partial}{\partial t}\ln\frac{1}{|t-x|} + K_2(t,x)\right)\varphi_2(t)dt = f_2(x); \quad x \in (-1,1) \tag{5.4}$$

Here

$$\varphi_2(t) = a_u(\gamma_+ - \gamma_- t), \quad K_2(t,x) = \gamma_-^2 b\int_{-1}^{x}\tilde{g}_{12}(\gamma_-(\xi - t))d\xi - \frac{\partial}{\partial t}\ln\frac{1}{|t-x|}$$

$$\tilde{g}_{ij}(t) = \sqrt{\frac{\pi}{2}}\left(\frac{4}{k_2^2|t|}\left(k_j^3 H_1^{(2)}(k_j|t|) - k_i^3 H_1^{(2)}(k_i|t|) + \right.\right.$$

$$\left.\left. + (-1)^i\frac{3}{|t|}(k_2^2 H_2^{(1)}(k_2|t|) - k_1^2 H_2^{(1)}(k_1|t|))\right) - k_2^2 H_0^{(2)}(k_j|t|)\right), \quad i,j = 1,2 \tag{5.5}$$

$$f_2(x) = \gamma - c\tau_{xz}^0(\gamma_+ - \gamma_- x), \quad b = -\sqrt{\frac{\pi}{2}}\frac{\lambda + 2\mu}{2(\lambda+\mu)}, \quad c = i\frac{\pi(\lambda+2\mu)}{\mu(\lambda+\mu)}$$

and $H_n^{(1)}(\cdot)$ is the Hankel function of the first kind and order n.

The same observations apply to the numerical method of solving this hypersingular integral equation as applied in the case of singular integral equation (5.1). In the Bubnov–Galerkin method, Chebyshev polynomials of the second kind are used as the basis and trial functions.

In Fig. 1 curve 3 represents the approximate solution of integral equation (5.4). The initial data is the same as in the case of integral equation (5.1). For convenience the required function is denoted in the figure by $\varphi_1(\cdot)$.

Consider integral equation (4.6). The principal part of the operator of the left-hand side of integral equation (4.6) is the same as in the case of integral equation (4.5).

The operator of the left-hand side of integral equation (4.6), like integral equation (4.5), is a bounded Fredholm pseudo-differential operator of order +1 and index 0, acting from $\tilde{H}^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. Then, it can be shown [7], that for any $a_u(\cdot), a_v(\cdot) \in \tilde{H}^{1/2}(\Gamma)$, the functions $u(\cdot,\cdot), v(\cdot,\cdot) \in H_{\text{loc}}^1(R^2)$. Hence, the displacements $u(\cdot,\cdot), v(\cdot,\cdot)$ satisfy the "condition on the edge", and the dynamic problem for a plane with a crack is uniquely solvable [6].

It is convenient to use the following equality for the inverse Fourier transformation of the kernel of the operator of the left-hand side of integral equation (4.6)

$$\frac{\xi^2}{\gamma_1(\xi)} = \frac{k_1^2}{\gamma_1(\xi)} - \gamma_1(\xi)$$

It can be shown that integral equation (4.6) is equivalent to the following integral equation

$$\frac{\partial}{\partial x}\int_{\alpha}^{\beta}\frac{\partial}{\partial t}\ln\frac{1}{|t-x|}a_v(t)dt + \frac{\partial}{\partial x}\int_{\alpha}^{\beta}\left(b\int_{\alpha}^{x}\tilde{g}_{21}(t-\xi)d\xi - \frac{\partial}{\partial t}\ln\frac{1}{|t-x|}\right)a_v(t)dt =$$

$$= -c\sigma_z^0(x); \quad x \in (\alpha,\beta) \tag{5.6}$$

The function $\tilde{g}_{21}(\cdot)$ and the constants b and c are defined by the third and the last two formulae of (5.5).

The approximate solution of the integral equation, to which integral equation (5.6) reduces on changing from the interval (α, β) to the interval $(-1, 1)$, is similar to the solutions shown in the figure by curves 3.

Hence, we have considered hypersingular integral equations, equivalent to the dynamic problem for a plane with a defect.

Note that in the dynamic problems considered, the longitudinal and transverse potentials are auxiliary. They can be conveniently used, since it is much easier to apply a Fourier transformation to the independent Helmholtz equations than to the connected Lamé equations. Moreover, in the auxiliary problem we can distinguish the problems for the Fourier transforms of longitudinal and transverse potentials. In this case the boundary integral equations are obtained in terms of displacements and stresses, and not potentials. Hence, the use of the functions $\varphi(\cdot, \cdot)$, $\psi(\cdot, \cdot)$ does not reduce the smoothness of the solutions of the problems.

Note that in this paper, when solving the problems, a Fourier transformation is applied to all the variables. This immediately enables us to obtain algebraic equations for the Fourier transforms of the required functions instead of ordinary differential equations, which are obtained in the general approach to the solution of similar problems.

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